Technical Report NTU-IE-Chou-2003-T001

Demand Modeling with the Geometric Brownian Motion Process

Yi-Yu Liang 2003/3/5

The purpose of modeling demand in this study is to construct a model to provide testing data needed in research, rather than to forecast the demand. The testing data will support simulation experiments in research projects on capacity planning. This technical report also demonstrate the procedure of constructing a demand model with the geometric Brownian motion process, and the emphasis will be on parameter estimation and sample path generation, instead of demand analysis and comparison of different modeling approaches.

INTRODUCTION

Demand modeling entails the construction of a model to provide an experiment environment to describe demand dynamics, instead of forecasting it. It's not straightforward to measure the fitness of model. Nevertheless, the constructed model is expected to have "similar properties" of actual demand.

In the past decade, demand increases continuous with some unexpected shocks. Take the demand quantity of leading edge memory (LEM) ICs as an example (Figure 1), the upward trend is obvious and unexpected shocks can also be observed. The data is in millions of wafer and is adapted from Sematech (2002).

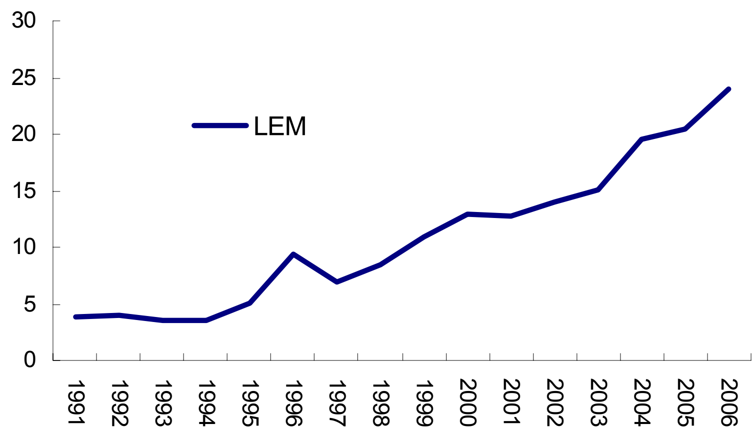

Figure 1 Demand dynamic of leading edge memory

For the time series of demand has an upward trend, rather than oscillation within a certain range, it is a non-stationary series. Unpredictable change is also an acknowledged characteristic of demand. Besides, that demand will have larger variance for farther time periods is also a reasonable description of demand dynamics.

Thus, geometric Brownian motion is suitable to model the dynamics of demand.

MODEL CONSTRUCTION

In the section, brief introduction to geometric Brownian motion will be given first, and the issue of parameter estimation will be explained next.

Geometric Brownian motion is usually used to describe the stock price in financial literature, such as Dixit and Pindyck (1994), Pindyck and Rubinfeld (1998), Tsay (2002). It is a continuous-time random walk series after logarithm transformation. The explanation of this property requires some derivations and it is put in the appendix. In recent years, Geometric Brownian motion has been used to describe the demand quantity in related literature of capacity planning (Benavides et al. 1999). To model demand with it, some introduction is given below. More detailed introductions should be referred to more advanced financial literature, such as Dixit and Pindyck (1994), Pindyck and Rubinfeld (1998), Tsay (2002).

Geometric Brownian motion is a special case of Brownian motion, or Wiener process. Starting from the equation of geometric Brownian motion, the variable q stands for demand quantity. The variable q follows geometric Brownian motion, if it satisfies the diffusion equation below,

$$dx_t = \mu x_t dt + \sigma x_t dw_t$$

Where $dw_t = \varepsilon_t \sqrt{dt}$ is the standard increment of Wiener process, and μ and σ is drift parameter and variance parameter respectively. ε_t has identical independent standard normal distribution.

With simple derivation below, the difference of q_t after logarithm transformation follows a normal distribution, and this implied q_t itself follows lognormal distribution.

$$\frac{dq_t}{q_t} = \mu dt + \sigma dw_t$$
$$d\ln(q_t) = (\mu - \sigma^2/2)dt + \sigma dw_t$$
$$\ln(q_T) - \ln(q_t) = \ln(\frac{q_T}{q_t}) \sim N[(\mu - \sigma^2/2)(T-t), \sigma^2(T-t)]$$

For simplicity, let r_t stand for the difference between q_t and q_{t-1} in finite interval after logarithm transformation as below.

$$\ln(\frac{q_t}{q_{t-1}}) = r_t$$

Let Δ be the given time interval between two observations. For example, Δ equals to one between q_t and q_{t-1}. The distribution of r_t follows a normal distribution with specified parameters,

$$r_t \sim N((\mu - \sigma^2/2)\Delta, \sigma^2\Delta) \qquad \ldots (1)$$

With the relations constructed above, the drift and variance parameters of geometric Brownian motion can be estimated with the sample mean $\bar{r}$ and standard error s_r from the data. Detailed derivations are listed below.

$$r_t = \ln(q_t) - \ln(q_{t-1})$$
$$E(r_t) = (\mu - \sigma^2/2)\Delta$$
$$V(r_t) = \sigma^2\Delta$$

$$\bar{r} = \frac{\sum_{t=1}^{n} r_t}{n}$$
$$s_r = \sqrt{\frac{1}{(n-1)}\sum_{t=1}^{n}(r_t - \bar{r})^2}$$

$$\hat{\sigma} = \frac{s_r}{\Delta}$$
$$\hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}$$

Having estimated the parameters, sample paths can be generated in either or two ways. The first one is to use the difference equation derived from the diffusion equation, which is

$$\frac{dq_t}{q_{t-1}} = \mu dt + \sigma dw_t = \mu dt + \sigma\varepsilon_t\sqrt{dt}$$
$$dq_t = \mu q_{t-1}dt + \sigma q_{t-1}dw_t$$
$$q_t - q_{t-1} = \mu q_{t-1}dt + \sigma q_{t-1}\varepsilon_t\sqrt{dt}$$
$$q_t = (1 + \mu dt)q_{t-1} + (\sigma\sqrt{dt})q_{t-1}\varepsilon_t, \text{ where } \varepsilon_t \sim N(0,1)\ i.i.d$$

Alternatively, data can be generated with the results derived with Ito's lemma. This result takes some stochastic calculus to derive. You can check other financial literature, such as Tsay (2002) for more details.

$$\ln(q_t) = \ln(q_0) + (\mu - \frac{\sigma^2}{2})t + \sigma w_t$$

$$q_t = q_0 \exp[(\mu - \frac{\sigma^2}{2})t + \sigma w_t] = q_0 \exp[(\mu - \frac{\sigma^2}{2})t + \varepsilon_t \sigma \sqrt{t}] \qquad \ldots (2)$$

As we can observe from the equation above, the distribution of q_t will have higher variance for farther time point *t,* since it is positive related with *t.*

The second one is a better approach to simulate the demand dynamics, since it's derived with Ito's lemma. To sample enough observation from ε_t, the random number generator of S-Plus, or RAND and NORMINV functions of Microsoft Excel can be used. Note that dt stands for the chosen time interval of the sample paths.

NUMERICAL EXAMPLE

With demand data of leading edge memory, the sample mean and standard error are derived first.

$$\bar{r} = 0.1230$$
$$s_r = 0.2137$$

The estimated drift and variance parameter of geometric Brownian motion can be further derived with sample mean and standard error. Here, the parameter $\Delta = 1$ for annual data.

$$\hat{\sigma} = \frac{s_r}{1} = 0.2137$$
$$\hat{\mu} = \frac{\bar{r}}{1} + \frac{\hat{\sigma}^2}{2} = 0.1459$$

Finally, the approximated series equation is constructed in the formulation derived above, and four sample paths are generated from the equation below. As observed, geometric Brownian motion has reasonable performance for the data.

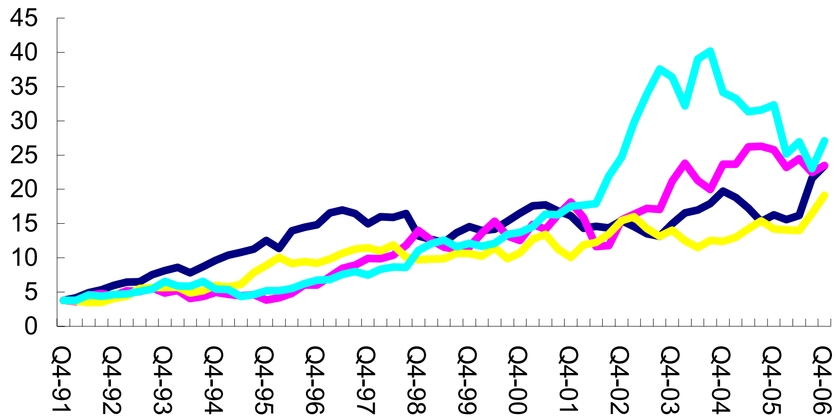

Figure 2 Simulated sample paths

The statistical hypothesis tests to verify the suitability of both original and simulated data are not included in the document. However, they have been done and the claim that demand follows geometric Brownian motion is acceptable. More issues about those hypothesis tests can refer to econometric or financial literature, such as Pindyck and Rubinfeld (1998), and Tsay (2002). S-Plus doesn't provide Dickey-Fuller test, which is relevant to test the suitability of original data, and it should be done with its add-on S+Finmetrics, or other statistical software. For example, Stata is suited to conduct those needed test of time series analysis.

FUTUER IMPLEMENTATION

This is a preliminary study of demand modeling with geometric Brownian motion, and some issues can be further implemented in the future. Within the domain of geometric Brownian motion, the estimators of drift and variance parameters can be modified, since there will be better estimators. Besides that, the series equation derived can be also revised, and it could be made more precise. Geometric Brownian motion may be sufficient to meet our needs. However, there are other diffusion equations in scientific research. These implementations need strong quantitative abilities in probability and statistics, and some experience and differential equations.

APPENDIX

In the last section, the statement "geometric Brownian motion is a continuous-time random walk series after logarithm transformation" is verified through some not-strict derivations as following.

x_t is said to be a time series with random-walk form if

$$x_t = \varepsilon_t + x_t$$

Starting from the diffusion equation of geometric Brownian motion,

$$\frac{dq_t}{q_{t-1}} = \mu dt + \sigma dw_t = \mu dt + \sigma \varepsilon_t \sqrt{dt}$$

$$dq_t = \mu q_{t-1} dt + \sigma q_{t-1} dz$$

$$q_t - q_{t-1} = \alpha q_{t-1} dt + \sigma q_{t-1} \varepsilon_t \sqrt{dt}$$

$$q_t = (1 + \mu dt) q_{t-1} + (\sigma \sqrt{dt}) q_{t-1} \varepsilon_t$$

$$q_t = \phi_1 q_{t-1} + \phi_2 q_{t-1} \varepsilon_t$$

$$q_t^* = \ln(q_t) = \ln(\phi_1 q_{t-1} + \phi_2 q_{t-1} \varepsilon_t) = \ln(\phi_1 + \phi_2 \varepsilon_t) + \ln(q_{t-1})$$

$$q_t^* = \varepsilon_t^* + q_{t-1}^*$$

Thus, the statement is verified since it has a random-walk form. (In fact, it should take additional manipulation to make sure that new term of white noise has zero mean.)

REFERENCE

1. Benavides, Dario L James R. Duley, and Blake E. Johnson, “As Good as It Gets: Optimal Fab Design and Deployment,” IEEE Transactions on Semiconductor Manufacturing, Vol. 12, No. 3, 281-287, 1999.
2. Pindyck, Robert S. and Daniel Rubinfeld, *Econometric Models and Economic Forecasts*, 4/e, McGraw-Hill, 1998.
3. Anonymous, *Industrial Economic Model User Guide,* Version 5.01, Sematech International, 2002
4. Dixit, Avanish K. and Robert S. Pindyck, *Investment under Uncertainty*, Princeton University Press, 1994.
5. Tsay, Ruey S., *Analysis of Financial Time Series,* New York, United States of America, John Wiley and Sons, Inc., 2002.